

Gravitational Atom in Compactified Extra Dimensions

E.G. Floratos^{a,b}, G.K. Leontaris^c and N.D. Vlachos^d

^a*Physics Department, University of Athens, Zografou 157 84 Athens, Greece*

^b*Institute of Nuclear Physics, NCSR Demokritos, 15310, Athens, Greece*

^c*Theoretical Physics Division, Ioannina University, GR-45110 Ioannina, Greece*

^d*Theoretical Physics Division, Aristotle University, GR-54006 Thessaloniki, Greece*

Abstract

We consider quantum mechanical effects of the modified Newtonian potential in the presence of extra compactified dimensions. We develop a method to solve the resulting Schrödinger equation and determine the energy shifts caused by the Yukawa-type corrections of the potential. We comment on the possibility of detecting the modified gravitational bound state Energy spectrum by present day and future experiments.

1. Introduction

Over the last few decades considerable experimental work has been devoted to test the accuracy of Newton's Gravitational Inverse Square Law (ISL) at short distances. To that end, a number of experiments using various sophisticated devices were designed to test the validity of ISL in distances as small as the sub-millimeter scale. Nowadays, one of the main theoretical motivations stimulating these extensive experimental searches is the prediction of Newton's Law modifications in theories with 'large' extra dimensions. Indeed, String Theory and related brane scenarios predict that our world is immersed in a higher 10-dimensional space where six of the ten dimensions are compact. A particular class of string constructions [1, 2] suggest that some of the extra dimensions could be decompactified at sub-millimeter distances, and manifest themselves through modifications of gravity and, in particular, of the Inverse Square Law. Recent experiments have tested the validity of ISL down to the scale of a few microns depending on the particular model and the experimental methodology [3]-[11].

Another class of experiments that have also been revived today measure quantum gravitational effects [12]-[16]¹. In order to avoid the dominance of electromagnetic interactions these experiments are performed with neutral particles. For instance, a neutron interferometer to measure the quantum mechanical phase shift of neutrons due to the interaction with Earth's gravitational field was proposed a long time ago in [13]. In recent experiments also, the quantum mechanical levels of a cold neutron beam above a flat optical mirror in Earth's gravitational field were also investigated [15, 16]. According to the predictions of models with extra-dimensions, modifications to Newton's law increase at shorter distances, in particular, close to or inside the compactification radius. Since the scale of quantum mechanical effects is many orders of magnitude smaller than the sub-millimeter scale -which is the range probed by the present experiments- possible modifications could become very important and eventually measurable at the atomic level.

In the experiments, the common parametrization of the corrections to the Newton's potential is considered to be of Yukawa type. Thus, the total potential is expressed as follows

$$\Phi(r) = -G_N \frac{M M'}{r} \left(1 + \alpha e^{-r/\lambda}\right). \quad (1)$$

The parameter α characterizes the strength of the Yukawa type correction to gravity, while λ accounts for the range of this extra interaction term. A considerable number of experiments testing the Newtonian nature of gravity have put strong limits [6, 10] on the strength and the range of the additional Yukawa interaction in (1).

Remarkably, it was found that the Yukawa-type correction in the above empirical formula (1) is of the same form with the leading correction term of the potential derived in the presence of extra compact dimensions. In the case of toroidal compactification in particular, it takes the form [18, 19]

$$\Phi(r) = -G_N \frac{M M'}{r} \left(1 + 2n e^{-r/R_C}\right) \quad (2)$$

where n is the number of extra dimensions and R_C the compactification radius. The radius R_C and the effective Planck scale M_C can be related as follows [2]: The Gauss' Law for distances $r \ll R$ results to the gravitational potential

$$\Phi(r) = - \left(\frac{\hbar}{c}\right)^n \frac{\hbar c}{M_C^{n+2}} \frac{M M'}{r^{n+1}}. \quad (3)$$

In the absence of extra dimensions, $n = 0$ and $M_C = M_P$ ² the above formula coincides with the standard four-dimensional Newton's potential.

For distances much larger than the compactification radius, $r \gg R_C$, we should recover the Newton's potential, and the formula takes the form

$$\Phi(r) = - \left(\frac{\hbar}{c}\right)^n \frac{\hbar c}{M_C^{n+2} R_C^n} \frac{M M'}{r}. \quad (4)$$

¹For recent experimental status see[17].

²It is to be mentioned that the Planck mass is expressed in terms of the gravitational constant as $M_P^2 = \frac{\hbar c}{G_N}$.

Comparing with (2)

$$M_C^{n+2} R_C^n = \frac{\hbar^2}{G_N} \left(\frac{\hbar}{c} \right)^{n-1} \quad (5)$$

which implies the following numerical relation

$$R_C = \left(\frac{\hbar G_N}{c^3} \right)^{\frac{1}{2}} \left(\frac{M_{Pl}}{M_C} \right)^{1+\frac{2}{n}} = 1.97 \times 10^{-17} e^{74.0821/n} \left(\frac{1\text{TeV}}{M_C} \right)^{1+\frac{2}{n}} \text{cm} \quad (6)$$

Given the number n of extra dimensions, formula (6) determines the radius as a function of the higher dimensional Planck scale. Thus, for one extra compact dimension, $n = 1$, a string scale as low as $M_C \sim 10\text{TeV}$, would lead to a ‘decompactified’ radius $R_C \sim 10^{10}$ meters, i.e. of the order of solar distances. For ranges up to this order, the Yukawa type correction in (2) is comparable to the ordinary gravitational term, implying observable hard violations to Newton’s law. However, the scale M_C is not determined by some principle and can be anywhere between M_W and M_P . It is observed that for $n = 1$ and compactification scale less than $\sim 10^9$ GeV, the compact radius is at most in the sub-millimeter range, thus, at distances $r \sim R_C$ corrections become important and would have been detected (for example, see relevant graphs in [6, 10]). However, for $M_C \geq 10^{10}\text{GeV}$ R_C drops down to 10^{-6}cm . In the presence of more than one compact dimensions, it is possible to considerably reduce the compactification scale without contradicting the present day experiments. Thus, as it can be checked from formula (6) for $n = 2$ and $M_C \sim 100\text{TeV}$ for example, we expect measurable modifications at distances $R \sim 10^{-7}\text{cm}$. This should be compared, for example, with the Bohr radius which is defined as

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e} = 5.29 \times 10^{-9}\text{cm} \quad (7)$$

The corresponding Bohr radius for a gravitational atom (gratom) containing a neutron instead of an electron could be defined as

$$a_G = \frac{\hbar^2}{G_N M_0 m_n^2} \quad (8)$$

where M_0 is the mass generating the gravitational potential.

From the above discussion, we see that experimental constraints restrict the $\lambda \sim R_c$ radius at minuscule distances where quantum mechanical effects might be sizable. This way, new experimental devices could possibly detect deviations from Newton’s law, or put more stringent bounds by means of appropriate quantum measurements. For example, in recent experiments, it has been shown that ultra-cold neutrons (UCN) in the Earth’s gravitational field form bound states. It turns out that consistency with Newton’s gravity is at the 10% level, so that bounds on non-standard gravity are put at the nanometer scale [16, 20, 21].

Motivated by the interesting results of the recent experimental activity, in this letter, we consider the quantum mechanical system of a ‘gravitational atom’ involving a light neutral elementary particle in the presence of extra compact dimensions. In particular, we study the corresponding Schrödinger equation that encodes the effects of the compact dimensions through a rather complicated modified Newton’s potential, aiming to obtain the modifications on measurable quantities³.

2. Gravitational Potential in the presence of extra compactified dimensions

In this section, we review in brief the derivation of the modified gravitational potential implied by the existence of an arbitrary number of extra compact dimensions and analyze its behavior at various distances. Then, we proceed to a mathematical analysis of the results and determine the behavior of the potential at various distances with respect to the radii of the compactified extra dimensions.

³Quantum mechanical effects from extra dimensions in various perspectives were studied also in references [22].

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ be vectors of the ordinary 3-dimensional space and x_i, y_i their corresponding coordinates. Assuming toroidal compactification, we denote $x_i^c = R\theta_i$, $i = 1, \dots, n$ the coordinates of the n compact dimensions with $\theta_i = [0, 2\pi)$ the corresponding angles while, for simplicity, we have adopted a common compactification radius R_C . In the presence of n compact extra dimensions the gravitational potential for two unit masses obeys the Laplace equation⁴

$$\nabla_{n+3}^2 \Phi = -\mu \delta^3(\vec{x} - \vec{y}) \frac{1}{R^n} \delta^n(\theta - \theta_0) \quad (9)$$

where for simplicity we introduced the parameter μ to account for various dimensional constants to be taken into account later on. Using the Fourier transform and performing the appropriate integrations in momentum space and restoring units, the solution is found to be

$$\Phi(r, \theta) = -G_N \frac{M M'}{r} \left(1 + 2 \sum_{\vec{m}} e^{-\frac{|\vec{m}|r}{R}} \cos(\vec{m} \cdot \vec{\theta}) \right). \quad (10)$$

where $r = |\vec{x} - \vec{y}|$, and the summation is over the tower of KK-modes in the dimensions of the compact space $\vec{m} = (m_1, m_2, \dots, m_n)$.

The first term in the potential (10) generates the standard gravitational inverse square law for the induced force. The second term is an infinite sum on KK-modes due to the presence of extra dimensions and describes a short range interaction exponentially suppressed by powers of $e^{-r/R}$. For distances much larger than the compactification radius however, (i.e. for $r \gg R$), all the terms of this infinite sum are highly suppressed by these exponential powers, thus (10) reduces to Newton's three dimensional analogue.

For measurements in the vicinity of the compactification radius $r \sim R$, the behavior of the infinite sum is not manifest. In the case of one extra dimension however, we may obtain an exact formula for the potential (10). Setting $\vec{m} = m$ and performing the sum for $n = 1$, we get

$$\Phi_{n=1}(r, \theta) = -G_N \frac{M M'}{r} \frac{e^{2r/R} - 1}{e^{2r/R} - 2e^{r/R} \cos \theta + 1}. \quad (11)$$

The effect of the compact dimensions is maximized for $\theta = 0$, where for the $n = 1$ case the potential assumes the simplified form

$$\Phi_{n=1}(r, 0) = -G_N \frac{M M'}{r} \coth\left(\frac{r}{2R}\right). \quad (12)$$

This formula interpolates between large and small distances r compared to the compactification scale R_C .

As already mentioned, for low compactification scales, M_C , taking $n = 1$ is unrealistic since it implies large corrections to the Newton's law at solar distances. Consequently, given the current experimental bounds [3]⁵ we have to imply either that there must be more than one large extra compact dimensions, or that the compactification scale is much smaller than a few microns. Nevertheless, from the last formula one can see that corrections near and below the compactification scale become substantially large and cannot be ignored.

The closed form derived for the case of one ($n = 1$) extra compact dimension [18]⁶, allows to determine the behavior of the corrected potential even inside the compact extra dimensional space where $r < R_C$, however, for $n > 1$ the sum as expressed in (10) cannot be performed. Instead, we may use the Jacobi transformation to express the potential as follows

$$\begin{aligned} \Phi(r, \theta) &\propto \frac{1}{(2\sqrt{\pi})^{n+3}} \int_0^\infty ds s^{-\frac{n+3}{2}} e^{-\frac{r^2}{4s}} \sum_{k=1}^n e^{-\frac{\theta_k R^2}{4s}} \\ &\times \left(1 + 2 \sum_{m_k} e^{-\frac{m_k \pi R}{s}} \cosh \frac{m_k \theta_k \pi R^2}{s} \right). \end{aligned} \quad (13)$$

⁴For convenience, from now on we drop the index C and simply write $R_C \rightarrow R$.

⁵For related bounds due to Casimir forces [23], see also [24].

⁶See also [25].

In order to examine the behavior of the potential, we first assume zero angles and perform the integration. For two extra dimensions the potential can be cast in the form $\Phi_{n=2} = \Phi_{n=1} + \Delta\Phi_{12}$ with [26]

$$\Delta\Phi_{12} = G_N \frac{M M'}{R} \sum_{n=-\infty}^{\infty} \frac{4}{\rho_n} \frac{d}{d\rho_n} \sum_{l=1}^{\infty} K_0(2\pi l \rho_n) \quad (14)$$

where $\rho = \frac{r}{2\pi R}$ and $\rho_n^2 = \rho^2 + n^2$. Numerical investigation shows that the quantity $\Phi_{n=1}$ is the main contribution to $\Phi(r, 0)$. Thus in the quantum problem the approximation (for the $\theta = 0$ case) of the $\Phi_{n=2}$ potential with $\Phi_{n=1}$ case is sufficient for our purposes and $\Delta\Phi_{12}$ can be ignored.

3. The Schrödinger equation in extra compactified dimensions

In this section, we seek solutions of the $(n+3)$ -dimensional Schrödinger equation with the modified gravitational potential $\Phi(r, \theta)$ discussed above

$$-\nabla_{n+3}^2 \Psi + \frac{2}{a_G} \Phi(r, \theta) \Psi = -\epsilon \Psi \quad (15)$$

where a_G has been defined in (8) while we have introduced the parameter

$$\epsilon = -\frac{2m_n E}{\hbar^2}. \quad (16)$$

For definiteness, here, we have taken m_n to be the neutron mass and M_0 in the a_G definition (8) to be some point like mass generating the potential. Our aim is to determine the wavefunctions and the energy levels in the presence of the modified gravitational potential. We will assume that the involved particles are neutral, so that electromagnetic potential terms, which would normally overwhelm any other source, are not present. In the subsequent analysis, we will concentrate on the case of one extra dimension only and introduce into Schrödinger's equation the potential (10) for $n = 1$.

We first observe that the modified gravitational potential exhibits an obvious 2π periodicity with respect to the parameter θ of the internal compact dimension. We find it useful to use an established transformation between the radial part of the Schrödinger's equation of angular momentum l to that of the isotropic oscillation in $2l+2$ dimensions [27]. This transformation, after the decoupling of the radial from the θ dependence, will help us to transform the problem into an equivalent system of coupled Hill-type equations of periodic potentials [26].

To start with, we parameterize the modified gravitational potential in the presence of compact extra dimensions as follows

$$\frac{2m\Phi(r, \theta)}{\hbar^2} \equiv -\frac{f(r, \theta)}{r} < 0. \quad (17)$$

In the case $n = 1$, the function $f(r, \theta)$ is derived from (11) to be

$$f(r, \theta) = \frac{g}{R} \left(1 + 2 \sum_{k=1}^{\infty} e^{-\frac{r}{R} k} \cos(k\theta) \right) \quad (18)$$

where

$$g \equiv \frac{2M_0 m_n^2}{M_C^3} = 2 \frac{R}{a_G}. \quad (19)$$

It should be noted that the new parameter g introduced in (18) is dimensionless.

Firstly, we should point out that the limit $\theta = 0$ and $r \rightarrow 0$, is singular. We may further clarify this point considering the case $n = 1$, where the potential is given by the closed formula (12). We observe that

the expansion for small r leads to the singular potential $\Phi(r) \sim \frac{1}{r^2}$. This is of course consistent with the fact that for distances much smaller than the compactification scale $r \ll R$ the potential assumes the familiar power law behavior (3). For weak couplings, however, as it is the case for the gravitational constant, the treatment of the $1/r^2$ potentials is quantum mechanically consistent. The failure of the potentials with higher singularities to produce a ground state is a well known fact which has been extensively discussed in the literature [28]. In the quantum mechanical treatment, the wavefunctions oscillate rapidly at the origin and there is no way to define a ground state. Nevertheless, away from the origin, a consistent description is still possible. This corresponds to looking at excited energy levels where the wavefunctions are less sensitive to the tower of KK states, which probe distances close to the origin.

The radial part of the Schrödinger equation has the familiar three dimensional form while the extra-dimensions dependence is encoded only in the potential through the function $f(r) \equiv f(r, 0)$:

$$\frac{d^2 \mathcal{R}}{dr^2} + \frac{2}{r} \frac{d\mathcal{R}}{dr} - \left(\epsilon - \frac{f(r)}{r} + \frac{l(l+1)}{r^2} \right) \mathcal{R} = 0. \quad (20)$$

This equation is to be trusted for distances higher than the compactification scale $r > R_C$.

We now apply the transformation $r \rightarrow \frac{z^2}{2\sqrt{\epsilon}}$, $\mathcal{R}(r) \rightarrow \frac{p(z)}{(4\epsilon)^{7/8} z^{3/2}}$ to get

$$\frac{d^2 p}{dz^2} - \left[z^2 + \frac{(1+4l)(3+4l)}{4z^2} - \frac{2}{\sqrt{\epsilon}} f\left(\frac{z^2}{2\sqrt{\epsilon}}\right) \right] p = 0 \quad (21)$$

with the new parameter z being dimensionless. Upon defining a new parameter $a = \frac{1}{2}(1+4l)$ it is observed that equation (21) is a generalized form of the radial part of the Schrödinger's equation of angular momentum l to that of the isotropic oscillation in $2l+2$ dimensions. It is easy to check that for the Newton's potential ($f(r) = 1$) the energy levels are given by $\epsilon_n = \frac{g^2}{4R^2(n+l+1)^2}$ and the principal quantum number is $N = n + l + 1$.

A general analysis for all values of l will be considered elsewhere [26]. Here, we will only consider the case $l = 0$ which corresponds to the value $a = 1/2$, so that (21) reduces to

$$\frac{d^2 p}{dz^2} - \left[z^2 + \frac{3}{4z^2} - \frac{2}{\sqrt{\epsilon}} f\left(\frac{z^2}{2\sqrt{\epsilon}}\right) \right] p = 0. \quad (22)$$

For $f(r) = 1$ the differential equation (22) is reduced to the known simple case [28] whose solutions are given in terms of the Laguerre functions,

$$u_{n, \frac{1}{2}}(z) = z^{\frac{3}{2}} e^{-\frac{1}{2}z^2} {}_1F_1(-n, 2, z^2) = \frac{1}{n+1} z^{\frac{3}{2}} e^{-\frac{1}{2}z^2} L_n^1(z^2).$$

We now express the solution $p(z)$ of (22) as a functional series in the eigenfunction basis of the $f(r) = 1$ equation, where the expansion coefficients are to be determined

$$p(z) = \sum_{n=0}^{\infty} c_n u_{n, \frac{1}{2}}(z). \quad (23)$$

Using the orthogonality properties of $L_n^1(z^2)$, we get the condition

$$\int_0^{\infty} u_{n, \frac{1}{2}}(z) u_{m, \frac{1}{2}}(z) dz = \frac{\delta_{mn}}{2(n+1)}. \quad (24)$$

Substituting the series (23) into (22) and multiplying with $u_{m, a}(z)$, integration over z gives

$$\sum_{n=0}^{\infty} c_n \left[\delta_{mn} - \frac{1}{\sqrt{\epsilon}} \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) f\left(\frac{z^2}{2\sqrt{\epsilon}}\right) dz \right] = 0.$$

In order to have a solution, the determinant of the above equation must vanish. This vanishing determines the energy eigenvalues as well as the expansion coefficients c_n in (23).⁷

It is now straightforward to consider effects introduced by adding one extra dimension. To this end, we include the second-order derivative in the Laplacian for the extra compact dimension $x_c = R\theta$, while we restore the θ -dependence in the potential. We expand now $p(z, \theta)$ in the same eigenfunction basis $u_{n, \frac{1}{2}}$, but in terms of θ -dependent coefficients $c_n(\theta)$:

$$p(z, \theta) = \sum_{n=0}^{\infty} c_n(\theta) u_{n, \frac{1}{2}}(z). \quad (25)$$

We substitute the series into the Schrödinger equation and multiply with $u_{m, \frac{1}{2}}(z)$. Finally, we integrate as previously over z and we end up with a system of coupled differential equations for the θ -variable dependent expansion coefficients c_n :

$$\sum_{n=0}^{\infty} \frac{1}{R^2 \epsilon} \frac{d^2 c_n(\theta)}{d\theta^2} \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) z^2 dz + \left[2 - \frac{2}{\sqrt{\epsilon}} \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) f\left(\frac{z^2}{2\sqrt{\epsilon}}, \theta\right) dz \right] c_n(\theta) = 0. \quad (26)$$

For the first integral over the z -variable we find [29]

$$A_{mn} = \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) z^2 dz = \frac{(-1)^{m+n} \sin \pi(m-n)}{\pi(m-n) [1 - (m-n)^2]}.$$

For $m = n$, the integral is $A_{nn} = 1$. For the second integral we first introduce the dimensionless parameter $\alpha = \frac{1}{2R\sqrt{\epsilon}}$ and define

$$\alpha_k = \frac{k}{2R\sqrt{\epsilon}} \equiv k\alpha, \quad k = 1, 2, \dots \quad (27)$$

Then, the integral involving the function f is a sum over k of integrals of the form

$$B_{mn}(\alpha_k) = \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) e^{-\alpha_k z^2} dz = \int_0^{\infty} z^3 e^{-(1+\alpha_k)z^2} {}_1F_1(-m, 2, z^2) {}_1F_1(-n, 2, z^2) dz.$$

Using the relevant formula from [30], the coefficients B_{mn} are found to be

$$B_{mn}(\alpha_k) = \frac{\Gamma(m+n+2)}{2(n+1)!(m+1)!} \frac{\alpha^{n+m}}{(\alpha+1)^{m+n+2}} {}_2F_1\left(-m, -n; -m-n-1, \frac{\alpha_k^2-1}{\alpha_k^2}\right). \quad (28)$$

Now, all the coefficients in (26) are known thus, we have transformed the original Schrödinger equation into a system of an infinite number of Hill-type coupled Differential Equations for the $c_n(\theta)$'s.

4. The Energy Shifts

Let us now turn to the differential system (26). Our aim is to determine the energy shifts as well as the modified wavefunctions due to the presence of the additional potential terms escorting the unperturbed Newton's potential. Because the extra terms lead to an infinite number of coupled differential equations, we naturally expect that the shift of any energy level will depend on the infinite tower of the energy levels of the unperturbed equation. It is further expected that the individual energy levels due to Bloch's theorem will turn to energy bands.

As we have already said, in this work we elaborate on the case $\theta = 0$ where we expect the effects to be maximal. Substituting the relevant form for the gravitational function $f(r, 0)$ into (26), the expansion coefficients $c_n(0)$ satisfy

$$\sum_{n=0}^{\infty} \left[\delta_{mn} - \frac{1}{\sqrt{\epsilon}} \frac{g}{R} \int_0^{\infty} u_{m, \frac{1}{2}}(z) u_{n, \frac{1}{2}}(z) \left(1 + 2 \sum_{k=1}^{\infty} e^{-\alpha_k z^2} \right) dz \right] c_n(0) = 0.$$

⁷ Notice that in the case of $f(r) = 1$ in particular, using (24), we simply recover the Balmer formula.

Performing the integrations while defining

$$D_{mn}(\alpha) = 2 \sum_{k=1}^{\infty} B_{mn}(\alpha_k),$$

it follows that the above system reduces into a linear system of equations

$$\sum_{n=0}^{\infty} \left[\left(R\sqrt{\epsilon} - \frac{g}{2(n+1)} \right) \delta_{mn} - g D_{mn}(\alpha) \right] c_n(0) = 0. \quad (29)$$

In this simplified form, we can easily observe that if the coefficients D_{mn} are set equal to zero, we immediately obtain the standard ‘Coulombic’ energy levels $\epsilon_n = \frac{g^2}{4R^2} \frac{1}{(n+1)^2}$. Thus, our task is to find the modifications implied by the presence of the D_{mn} contributions. Due to the non-diagonal form of the latter it can be easily deduced that any energy level ϵ_n receives corrections from an infinite number of energy levels. Practically, we aim to find the dimensionless eigenvalue $R\sqrt{\epsilon_n}$ in terms of the dimensionless coupling $g = \frac{2M_0 m^2}{M_C^2} = 2 \frac{R_C}{a_G}$ using a finite but adequately large number N of states in (29). It should be noted that this truncation does not presume that the coupling g in front of the correction terms D_{mn} is small. As a matter of fact, it is expected that higher n -states will contribute less, so a sufficiently large number N in the sum (29) will lead to a stable result. The problem then is transformed to a $N \times N$ -matrix equation where we are seeking solutions for the eigenvectors $\vec{c}(\theta) = (c_1, c_2, \dots, c_N)^T$ and their corresponding energy eigenvalues ϵ_n . To further proceed, we use the definition (27) to write

$$\sum_{n=0}^{\infty} \left[\left(\frac{1}{\alpha} - \frac{g}{(n+1)} \right) \delta_{mn} - 2g D_{mn}(\alpha) \right] c_n(0) = 0.$$

The energy levels are then given by the solutions of the equation

$$\left| \left(\frac{1}{\alpha} - \frac{g}{(n+1)} \right) \delta_{mn} - 2g D_{mn}(\alpha) \right| = 0.$$

As noted, the above method works, even if g is not in the perturbative region. However, for our present investigation, let us assume that g is small enough so that we can handle the quantities B_{mn} perturbatively. Considering that the perturbative term implies small corrections to the unperturbed eigenvalues, we seek solutions of the form

$$\frac{1}{\alpha_n} = \frac{g}{(n+1)} + \sum_{k=3}^{\infty} c_k^n g^k.$$

Omitting the calculational details (see [26]), we finally get the following result:

$$\alpha_n g = n + 1 - \frac{\pi^2}{3} g^2 + 4\zeta(3) g^3 + \frac{\pi^4}{9(1+n)} g^4.$$

It is interesting to note that each Coulombic energy level is shifted by a constant up to order g^3 since the first three expansion coefficients do not depend on the chosen dimensionality of D_{mn} . We can write the perturbed energy levels \mathcal{E}_n in terms of the Coulombic ones $E_n = \frac{g^2}{4R^2(n+1)^2}$ as follows

$$\mathcal{E}_n = \frac{E_n}{\left(1 - \frac{\pi^2}{3(n+1)} g^2 + \frac{4\zeta(3)}{n+1} g^3 + \frac{\pi^4}{9(1+n)^2} g^4 \right)^2} \quad (30)$$

In figure 1 we have plotted the correction to the gravitational energy levels for the first three values of the principal quantum number $n = 0, 1, 2$ and $l = 0$. We observe that for reasonable values of the coupling constant $g = 2 \frac{R_C}{a_G}$, between $\left[\frac{1}{10} - \frac{1}{4} \right]$, the corrections are experimentally detectable and are of the order of up to 15%. This means that if the compactification radius is $R_C \approx \left[\frac{1}{10} - \frac{1}{100} \right] a_G$, (where a_G the ‘Bohr’ radius of the gravitational atom), the corrections are sizable. The existing experiments measuring the

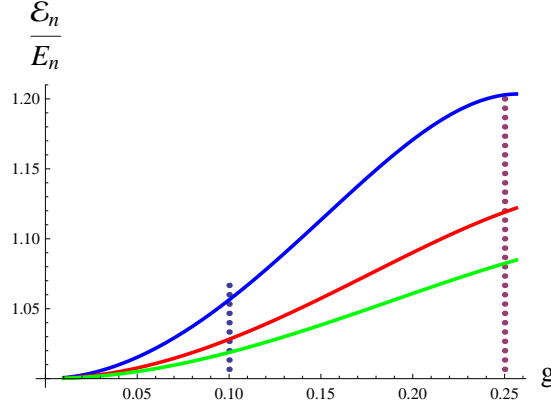


Figure 1: (Color on line) The ratios of the shifted energy levels over the Coulombic ones \mathcal{E}_n/E_n for the first three energy levels. The maximum deviation occurs at $g \sim \frac{1}{4}$ for the first Energy level (blue curve).

bound state energy spectrum of UCN beams on the Earth's gravitational potential use specific geometries of horizontal systems of reflectors and absorbers and essentially measure the energy levels by the distance of the absorber of the reflector and the flux of the outgoing neutrons. The extra dimensional corrections to the Newton's potential we are discussing here are negligible for this type of experiments. In our frame, we should have experiments of UCN beams and spherically symmetric high-density materials for which a gravitational radius a_G is larger than the compactification radius R_C and the radius of the spherical material r_M should lie between these two:

$$R_C < r_M < a_G . \quad (31)$$

To find if such materials exist in nature, we first recall the formula of the gravitational radius a_G of a spherical object of mass M_0 , given in (8). Using the numerical values of the universal constants we express the radius a_G in millimeters:

$$a_G = 59.4 \frac{1}{M_0/\text{gr}} \text{ mm} \quad (32)$$

In terms of the material density ρ_M and the radius of the spherical object inducing the gravitational potential, the radius a_G can be written

$$a_G = \frac{\hbar^2}{G_N m_n^2} \frac{3}{4\pi \rho_M r_M^3} . \quad (33)$$

Introducing the constant

$$\kappa = \frac{3\hbar^2}{4\pi G_N m_n^2} = 14.824 \text{ gr mm} \quad (34)$$

the inequality (31) gives the constraint for the density and the radius of the material

$$\frac{g}{2} < \frac{\rho_M r_M^4}{\kappa} < 1 . \quad (35)$$

We can put the above constraint in a more useful form

$$\left(\frac{g}{2}\right)^4 \frac{\kappa}{R_C^4} < \rho_M < \frac{g}{2} \frac{\kappa}{R_C^4} . \quad (36)$$

For $g \sim 0.2$, using a spherical device of the highest density material (Uranium), we can probe the extra dimension down to $R_C \sim 0.5$ mm. If we decrease g , we can probe smaller distances, but the perturbative energy shifts become tiny and rather hard to be detected by the experiment.

In figure 2 we draw the left-hand side (LHS) limit of the inequality (36) in the (ρ_M, R_C) -plane for two characteristic values of $g = \frac{1}{10}, \frac{1}{4}$ (for reasonable values of the expansion parameter g the right-hand side of (36) is experimentally irrelevant). For a given g -value the region in the (ρ_M, R_C) plane for which the R_C can be probed lies on the right of the corresponding g -curve. For convenience, we have also plotted the horizontal line $\rho_M = 0.019 \text{ gr/mm}^3$ which corresponds to the density of Uranium being the highest density material existing in Nature. Thus the probed R_C 's correspond to the region determined below this line and on the right to the g -curve. We observe that for the existing densities in nature and reasonable g -values the compactification radius R_C is above $\sim 0.2 \text{ mm}$ for the ground state while present day experiments constrain R_C to be smaller than $\sim 30 \text{ microns}$ for KK graviton scenarios of extra dimensions which is our case. To probe smaller radii in our diagram we could consider capturing the neutron into higher excited states $n = 1, 2, 3, \dots$. One could think other geometries of the gravitational source so to probe smaller compactification scales within the present gravity-modification scenario.

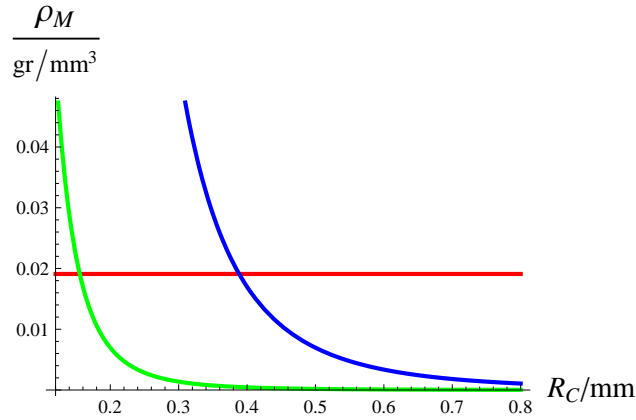


Figure 2: (Color on line) Plot of the LHS limit of inequality (36) for $g = \frac{1}{10}$ (lower curve) and $g = \frac{1}{4}$ (upper curve). The horizontal line corresponds to the density $19.1 \times 10^{-3} \text{ gr/mm}^3$ (Uranium). The corresponding energy shifts for the first three energy levels are found along the two vertical dotted lines in figure 1.

5. Conclusions

In this work we considered quantum gravitational effects produced by a modified gravitational potential from ‘decompactified’ extra-dimensions with radii R_C at the order of sub-micron scales. We calculated the energy levels of a hypothesized ‘gravitational atom’ formed by a neutron captured by a spherical mass. It was found that the energy-shifts ΔE_n , compared to the energy levels E_n of the unperturbed $\frac{1}{r}$ -potential, can be expressed in terms of simple powers of the perturbative expansion parameter $g = \frac{2R_C}{a_G}$ where a_G is the ‘Bohr’ radius of the ‘gravitational atom’.

We find that for reasonable values of the perturbative constant $g \sim [0.1 - 0.25]$, there are sizable $\Delta E_n \sim 10\%$ effects which are in principle measurable in properly designed experiments. However, stringent limits on the size of extra dimensions require either smaller g -values where ΔE_n effects start becoming negligible, or extremely dense materials to generate a ‘gravitational atom’ with sufficiently small a_G -radius. Probes with the simple spherical geometry considered in this simple analysis are not sufficient to generate such small radii. We envisage that more sophisticated geometries could be invented where these effects could be measured in future experimental explorations.

Acknowledgements The work of GKL and NDV is partially supported by the European Research and Training Network grant “Unification in the LHC era” (PITN-GA-2009-237920). The work of EGF is partially supported by the EKPA program Kapodistrias 70/4/9711. GKL and EGF would like to thank the Physics Theory group of École Normale Supérieure in Paris for kind hospitality during the last stage of this work.

References

- [1] I. Antoniadis, “A Possible new dimension at a few TeV,” *Phys. Lett. B* **246** (1990) 377.
- [2] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Phys. Lett. B* **429** (1998) 263 [arXiv:hep-ph/9803315].
- [3] J. C. Long, H. W. Chan and J. C. Price, “Experimental status of gravitational-strength forces in the sub-centimeter regime,” *Nucl. Phys. B* **539** (1999) 23 [arXiv:hep-ph/9805217].
- [4] E. Fischbach, D. E. Krause, V. M. Mostepanenko and M. Novello, “New constraints on ultrashort-ranged Yukawa interactions from atomic force microscopy,” *Phys. Rev. D* **64** (2001) 075010 [arXiv:hep-ph/0106331].
- [5] J. C. Long, H. W. Chan, A. B. Churnside, E. A. Gulbis, M. C. M. Varney and J. C. Price, “Upper limits to submillimeter-range forces from extra space-time dimensions,” *Nature* **421** (2003) 922.
- [6] J. C. Long and J. C. Price, “Current short-range tests of the gravitational inverse square law,” *Comptes Rendus Physique* **4** (2003) 337 [arXiv:hep-ph/0303057].
- [7] O. Bertolami and F. M. Nunes, “Ultracold neutrons, quantum effects of gravity and the weak equivalence principle,” *Class. Quant. Grav.* **20** (2003) L61 [arXiv:hep-ph/0204284].
- [8] E. G. Adelberger, B. R. Heckel and A. E. Nelson, “Tests of the gravitational inverse-square law,” *Ann. Rev. Nucl. Part. Sci.* **53** (2003) 77 [arXiv:hep-ph/0307284].
- [9] C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, “Sub-millimeter tests of the gravitational inverse-square law,” *Phys. Rev. D* **70** (2004) 042004 [arXiv:hep-ph/0405262].
- [10] A. A. Geraci, S. J. Smullin, D. M. Weld, J. Chiaverini and A. Kapitulnik, “Improved constraints on non-Newtonian forces at 10 microns,” *Phys. Rev. D* **78** (2008) 022002 [arXiv:0802.2350 [hep-ex]].
- [11] R. S. Decca, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, D. Lopez and V. M. Mostepanenko, “Application of the proximity force approximation to gravitational and Yukawa-type forces,” *Phys. Rev. D* **79** (2009) 124021 [arXiv:0903.1299 [quant-ph]].
- [12] G. van der Zouw, M. Weber, A. Zeilinger, J. Felber, R. Gaehler and P. Geltenbort, “Aharonov-Bohm and gravity experiments with the very-cold-neutron interferometer,” *Nucl. Instrum. Meth. A* **440** (2000) 568.
- [13] R. Colella, A. W. Overhauser and S. A. Werner, “Observation of gravitationally induced quantum interference,” *Phys. Rev. Lett.* **34** (1975) 1472.
- [14] V. I. Lushchikov and A. I. Frank, “Quantum Effects At The Storage Of Ultracold Neutrons On A Plane,” *Phys. Rev.D* **66** (2002) 010001.
- [15] V. V. Nesvizhevsky *et al.*, “Quantum states of neutrons in the Earth’s gravitational field,” *Nature* **415** (2002) 297.
- [16] V. V. Nesvizhevsky and K. V. Protasov, “Constraints on non-Newtonian gravity from the experiment on neutron quantum states in the earth’s gravitational field,” *Phys. Rev.D* **66** (2002) 010001 [arXiv:hep-ph/0401179].
- [17] I. Antoniadis, “Physics and signatures of strings and extra dimensions,” *Int. J. Mod. Phys. E* **16** (2007) 2733.
“GRANIT-2010” Conference, Les Houches, 14-19 February 2010,
<http://lpsc.in2p3.fr/Indico/conferenceDisplay.py?confId=371>

- [18] E. G. Floratos and G. K. Leontaris, “Low scale unification, Newton’s law and extra dimensions,” *Phys. Lett. B* **465** (1999) 95 [arXiv:hep-ph/9906238].
- [19] A. Kehagias and K. Sfetsos, “Deviations from the $1/r^2$ Newton law due to extra dimensions,” *Phys. Lett. B* **472** (2000) 39 [arXiv:hep-ph/9905417].
- [20] A. Westphal, H. Abele and S. Baessler, “Analytically derived limits on short-range fifth forces from quantum states of neutrons in the earth’s gravitational field,” arXiv:hep-ph/0703108.
- [21] S. Dimopoulos and A. A. Geraci, “Probing sub-micron forces by interferometry of Bose-Einstein condensed atoms,” *Phys. Rev. D* **68** (2003) 124021 [arXiv:hep-ph/0306168].
- [22] A. Brandhuber and K. Sfetsos, “Non-standard compactifications with mass gaps and Newton’s law,” *JHEP* **9910** (1999) 013 [arXiv:hep-th/9908116].
H. w. Yu and L. H. Ford, ‘Lightcone fluctuations in quantum gravity and extra dimensions,’ *Phys. Lett. B* **496** (2000) 107 [arXiv:gr-qc/9907037].
S. Hossenfelder, “Running coupling with minimal length,” *Phys. Rev. D* **70** (2004) 105003 [arXiv:hep-ph/0405127].
- [23] M. Bordag, U. Mohideen and V. M. Mostepanenko, “New developments in the Casimir effect,” *Phys. Rept.* **353** (2001) 1 [arXiv:quant-ph/0106045].
M. Bordag, G. L. Klimchitskaya, U. Mohideen and V. M. Mostepanenko, ‘Advances in the Casimir effect,’ *Int. Ser. Monogr. Phys.* **145**, 1 (2009).
- [24] V. M. Mostepanenko and M. Novello, “Constraints on non-Newtonian gravity from the Casimir force measurements between two crossed cylinders,” *Phys. Rev. D* **63** (2001) 115003 [arXiv:hep-ph/0101306].
V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko and C. Romero, “Advance and prospects in constraining the Yukawa-type corrections to Newtonian gravity from the Casimir effect,” *Phys. Rev. D* **81** (2010) 055003 [arXiv:1002.2141 [Unknown]].
R. S. Decca, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, D. L. Lopez and V. M. Mostepanenko, “Improved tests of extra-dimensional physics and thermal quantum field theory from new Casimir force measurements,” *Phys. Rev. D* **68** (2003) 116003 [arXiv:hep-ph/0310157].
- [25] V. K. Oikonomou, “Corrections to Gravity due to a Sol Manifold Extra Dimensional Space,” *Class. Quant. Grav.* **25** (2008) 195020 [arXiv:0801.3527 [hep-th]].
M. Bures, “Atoms In Compactified Universes,” http://is.muni.cz/th/52540/prif_m/diplomka.pdf
- [26] E. G. Floratos, G. K. Leontaris, N. D. Vlachos, “In preparation”
- [27] D. Bergmann and Y. Frishman, “A Relation between the Hydrogen Atom and Multidimensional Harmonic Oscillators”, *J. Math. Phys.* **6**, 1855 (1965);
V. A. Kostelecky and N. Russell, “Radial Coulomb and Oscillator Systems in Arbitrary Dimensions,” *J. Math. Phys.* **37** (1996) 2166 [arXiv:quant-ph/9602007].
- [28] K. M. Case, “Singular potentials,” *Phys. Rev.* **80** (1950) 797.
W. Frank, D. J. Land and R. M. Spector, “Singular potentials,” *Rev. Mod. Phys.* **43** (1971) 36.
K.K. Singh, “Energy levels of the bounded isotropic harmonic oscillator and the bounded hydrogen atom ” *Inst. Sci. India.*; Vol: Pt. A, 27.
D. Bergmann and Y. Frishman, “A Relation between the Hydrogen Atom and Multidimensional Harmonic Oscillators,” *J. Math. Phys.* **6** (1965) 1855
- [29] Yuri Aleksandrovich Brychkov, Anatolii Platonovich Prudnikov, “Integral Transforms of Generalized Functions”, Gordon and Breach Science publishers, 1989.
- [30] I. S. Gradshteyn and I. M. Ryzhik, “Table of Integrals, Series, and Products” Alan Jeffrey and Daniel Zwillinger (eds.), Seventh edition (Feb 2007).
- [31] E. T. Whittaker and G. N. Watson, “A Course of Modern Analysis”, Cambridge University Press.